**Introduction**

In a rotating frame of reference objects move with an additional force usually referred to as a fictitious force. For frames rotating at a constant angular velocity $\omega$ a Centripetal and Coriolis force is present and can be described by

$$F_{\text{cent}}(\vec{r}) = m\omega^2\vec{r}_\perp$$  \hspace{1cm} (1a)$$

$$\vec{F}_{\text{cor}}(\vec{v}) = -2m(\vec{\omega} \times \vec{v})$$  \hspace{1cm} (1b)$$

With $\vec{r}_\perp$ is the position perpendicular to $\omega$. It is the objective of this piece to develop a space-time description of a rotating frame.

**Potential Energy for a Rotating Frame**

It can be shown using Lagrangian mechanics that an object in a rotating frame behaves as if its potential energy is due to a centripetal and Coriolis force.

$$U_{\text{cent}}(\vec{r}) = -m\frac{\omega^2}{2}r_\perp^2$$  \hspace{1cm} (2a)$$

$$U_{\text{cor}}(\vec{r},\vec{v}) = 2m\omega \cdot (\vec{\omega} \times \vec{r})_\omega$$  \hspace{1cm} (2b)$$

Suppose we choose as a coordinate system a cylindrical coordinate system $(r,\theta,z)$ with $\theta$ about the z-axis. Under these conditions the potential energy terms in (2) combine to become

$$U_{\text{rot}}(\vec{r}) = -m\frac{(r\omega)^2}{2} + 2m\omega \cdot r \cdot v_\theta$$

$$U_{\text{rot}}(\vec{r}) = -m\frac{(r\omega)^2}{2} + 2 \omega \cdot r \cdot P_\theta$$  \hspace{1cm} (3)$$

The relativistic mass times the potential energy for this case (remembering that $P_t = imc$) gives

$$m \cdot U_{\text{rot}} = \left(\frac{2\omega r}{ic}\right) \cdot (P_\theta P_t) - \frac{1}{2} \cdot \left(\frac{r\omega}{c}\right)^2 \cdot P_t^2$$  \hspace{1cm} (4)$$

1.
Rotating frame scale factors

An observer in a rotating frame could account for the behavior of objects under the potential energy (3) by the geometry of the space-time of the region. The next task therefore is to bring in the theory of generalized gravity\(^2\) to obtain the geometry of space-time in a rotating frame. To do so, rules for using the theory of generalized gravity must be devised.

Symmetry and scale factors

If the physical source of a force on objects has any symmetry, then that symmetry should be manifested in the geometry of the region. This suggests the follows rules:

1. If the source of a potential energy is symmetric about some \(x_\alpha\) (i.e. symmetric under the transformation \(x_\alpha \rightarrow -x_\alpha\) or \(x_\alpha \rightarrow x_\alpha + \delta x_\alpha\)), then the deviation of the scale factors from the unperturbed geometry must be independent of \(x_\alpha\).
2. There must be a consistency between observers. For example any two points along a trajectory of a beam of light will have \(\Delta s = 0\).
3. If there is no are no other restrictions, \(h \rightarrow L\).

The last restraint must be employed with care.

The scale factors

For the case of a rotating frame with constant \(\omega\), there is no dependence on \(\theta, t\) or \(z\) (assuming the axis of rotation is along the \(z\)-axis). The scale factors therefore must depend only on \(r\). Moreover the theory of generalized gravity dictates that, because of the Coriolis force potential energy, there must be a non-zero \(h_{\theta\theta}\) and \(h_{tt}\) term that should be equal to each other. The scale factor can now be known to have the form

\[
 ds^2 = (h_r dr)^2 + (dz)^2 + (h_\theta d\theta)^2 + (2h_{t\theta})(dt \cdot d\theta) + (h_t dt)^2 \tag{5}
\]

Solving for \(h_\theta, h_t\), and \(h_{t\theta}\)

The equations for for \(h_\theta, h_t\), and \(h_{t\theta}\) are below, obtained from the generalized gravity equations for \(h_\theta\) and \(h_{t\theta}\) respectively.

\[
 2 \frac{2h_t}{ic} = \left(\frac{r \omega}{c}\right)^2 \tag{6a}
\]
Borrowing from the Schwartzchild metric for time, postulate that $h_t$ has the general form

$$h_t = ic \cdot \sqrt{1 + A \cdot (r \omega)^2}$$  \hspace{1cm} (7)$$

Remembering that an object can be said to conserve energy only when $h_\alpha \approx L_\alpha$, this must occur in this case for small $r$. We will make the following approximation for the calculations.

$$h_t \approx ic \left(1 + \frac{1}{2} A \cdot (r \omega)^2\right)$$  \hspace{1cm} (8)$$

Equation (5a) now reads

$$2 - 2 \cdot \left(1 + \frac{A}{2} \cdot (r \omega)^2\right) = \left(\frac{r \omega}{c}\right)^2$$  \hspace{1cm} (9)$$

$$A = -\frac{1}{c^2}$$  \hspace{1cm} (10)$$

The general time scale factor is

$$h_t = ic \cdot \sqrt{1 - \left(\frac{r \omega}{c}\right)^2} = ic \cdot \sqrt{1 - \left(\frac{r}{r'}\right)^2}$$  \hspace{1cm} (11a)$$

$$r' = \frac{c}{\omega}$$  \hspace{1cm} (11b)$$

Next, substitute $h_t$ into (5b). Moreover use the approximation for $h_t$ appropriate to the limit $r \ll r'$.

$$h^2_{t_0} = \left(\frac{2 \omega r}{ic}\right) \cdot (1 - \frac{1}{2} \left(\frac{r}{r'}\right)^2) \cdot (ich_\theta)$$  \hspace{1cm} (12)$$

For the case of $r \ll r'$, $h_\theta = L_\theta = r$.

$$h^2_{t_0} = 2 \omega r^2$$  \hspace{1cm} (13)$$

This would be true at least for $r \ll r'$. Expanding to larger $r$ requires one to consider speeds at which a beam of light travels if it does so in the $x_k$ direction.

**The speed of light**

Insisting that $ds^2 = 0$ for any two points along the path of light gives us a constraint to work with. Assuming that a light beam sent along the $x_k$ direction.

3.
\[ 0 = h_k^2 \cdot dx^2 + h_k^2 \cdot dx_k^i \cdot dt + h_i^2 \cdot dt^2 \]  
(13)

Divide by \( dt^2 \).

\[ 0 = h_k^2 \cdot \left( \frac{dx_k^i}{dt} \right)^2 + h_k^2 \cdot \left( \frac{dx_k^i}{dt} \right) + h_i^2 \]  
(14)

The derivatives can be converted into speeds for the light beam by including scale factors.

\[ 0 = \left( \frac{h_k}{L_k} \right)^2 \cdot (c_k')^2 + \left( \frac{h_k'}{L_k} \right) \cdot (c_k') + h_i^2 \]  
(15)

c_k' is the velocity if light in the \( x_k \) direction. The quadratic nature of (15) is consistent with the velocity of light having two directions.

For some region, the values \( c_k' \) indicates something about \( h_t, h_{kt} \) and \( h_k \).

- **Light can be at rest in a given region**: \( h_t \to 0 \).

- **Only one unique nonzero value of \( c_k' \) exists**: The quadratic term must vanish, so \( h_k \to 0 \).

- **Two nonzero values of \( c_k' \) exist**: Neither \( h_{kt} \) nor \( h_k \) must vanish, or go to \( \infty \) if \( h_t \to 0 \).

- **The solution to \( c_k' \) is zero and one nonzero value**: \( h_t \to 0 \) and \( h_{kt} \neq 0 \) and \( h_k \neq 0 \).

For the rotating case, as \( r \to r' \), \( c_\theta' \) must have two options. It can be zero (a non-rotating observer shoots a light beam in the direction of rotation) and some other nonzero value. The above rules indicate that in this limit \( h_t \to 0 \) (consistent with the above expression for \( h_t \)) and \( h_\theta \neq 0 \). Moreover we must insist that for \( r \ll r' \) \( c_\theta' \approx \pm c \), so \( h_\theta \to 0 \). The given value for \( h_\theta \) bears out the second condition. The first condition can also be satisfied if we assume \( h_\theta = L_\theta = r \) for all \( r \).

**Solving for \( h_z \) and \( h_i \)**

To determine \( h_i \) imagine the following: An observer is at rest in a rotating frame a distance \( r \) from its axis. Another observer is in an inertial frame and determines the first observer is a distance \( r_i \) from the axis and is moving at a speed \( v = r_i \cdot \omega \). For the case where \( r_i = r' \), \( v = c \). Any two events occurring at
the rotating observer will, by the estimation of the inertial frame observer, have \( \Delta s = 0 \), because our rule #2. So relative to this observer

\[
0 = -c^2 \left[ 1 - \left( \frac{r}{r'} \right)^2 \right] \delta t^2
\]  

(17)

For this to be true for all \( \delta t \), \( r \) must also be \( r' \), so the radial distances must be the same for both rotating and inertial observers.

\[
h_r = 1
\]  

(18)

So the space-time interval for a rotating frame is

\[
ds^2 = dr^2 + dz^2 + (rd\theta)^2 + (2\omega r^2)(dt-\frac{c}{y_r})^2
\]  

(19a)

\[
y_r = \sqrt{1 - \left( \frac{r}{r'} \right)^2}
\]  

(19b)

**Geodesics**

Suppose an object is in the space-time geometry as given above, what would its path be assuming it is a geodesic? In general relativity a geodesic trajectory is described by

\[
\frac{d^2 x_\gamma}{d\lambda^2} = -\sum_{\alpha, \beta} \Gamma_{\gamma \alpha \beta} \left( \frac{dx_\alpha}{d\lambda} \right) \left( \frac{dx_\beta}{d\lambda} \right)
\]  

(20)

The \( \Gamma \) is called the Christoffel Symbol and \( \lambda \) some parameter. The \( x \) coordinates span all coordinates and include the scale factors. The sum eliminates any duplicates in indices. The terminology of general relativity will be altered according to the following:

- Let the \( x_\alpha \) in general relativity be written as \( L_\alpha x_\alpha \).
- Let all derivatives of coordinates be rewritten as \( L_\alpha \frac{dx_\alpha}{d\lambda} \).
  Likewise for all higher order derivatives.
- Do not bother with contra vs. covariant quantities. If a given \( h_{\alpha\beta} \) has an ‘i’ in it, this will be taken care of.

With this in mind, the geodesic equation now is

\[
L_\gamma \frac{d^2 x_\gamma}{d\lambda^2} = -\sum_{\alpha, \beta} \Gamma_{\gamma \alpha \beta} \left( L_\alpha \frac{dx_\alpha}{d\lambda} \right) \left( L_\beta \frac{dx_\beta}{d\lambda} \right)
\]  

(21a)

And the Christoffel Symbol is now

5.
\[ \Gamma_{\alpha \beta \gamma} = L_\alpha \cdot \partial_\alpha \left[ \frac{h_{\beta \gamma}^2}{2 |L_\beta||L_\gamma|} \right] + L_\beta \cdot \partial_\beta \left[ \frac{h_{\alpha \gamma}^2}{2 |L_\alpha||L_\gamma|} \right] - L_\gamma \cdot \partial_\gamma \left[ \frac{h_{\alpha \beta}^2}{2 |L_\alpha||L_\beta|} \right] \] (21b)

For the case of the rotating frame, the possible nonzero values of \( \Gamma \) are (for \( r \ll \rho' \))

\[ \Gamma_{rr} = -\frac{r}{r' r^2} \] (22a)

\[ \Gamma_{\theta r} = -\frac{1}{r'} \] (22b)

\[ \Gamma_{r\theta} = \frac{r}{r' r^2} \] (22c)

\[ \Gamma_{r\theta} = \frac{1}{r'} \] (22d)

\[ \Gamma_{\theta r} = \frac{1}{r'} \] (22e)

In the classical limit, \( v/c \ll 1 \) and \( r \ll \rho' \), the geodesic trajectory obeys the following equations of motion.

\[ \frac{d^2 r}{dt^2} = \frac{r c^2}{r' r^2} + \left[ \frac{c}{r'} \right] \cdot v_0 = \omega^2 \cdot r + \omega \cdot v_0 \] (23a)

\[ r \cdot \frac{d^2 \theta}{dt^2} = \omega \cdot v_\theta \] (23b)

\[ c \cdot \frac{d^2 t'}{dt^2} = -\left( \frac{\omega \cdot v_\theta \cdot v_\phi}{c^2} \right) - \omega^2 \left( \frac{r' \cdot v_\rho}{c} \right) \to 0 \] (24c)

**Critique of geodesic**

- First term of (23a) is the centrifugal force term, the second term is part of the Coriolis force. It is needed so an object at rest in an inertial frame will move in a circular motion about the z axis with velocity \( v_0 = -\omega \cdot r \).
Equation (23b) says that an object “falling” in the \( r \) direction will be deflected in the \(-\theta\) direction. This is consistent with the Coriolis potential that favors motion in this direction (all assuming \( \omega \) is positive).

Equation (23c) is important in the classical limit for time to be the same for all observers. The proposed scale factors accurately describe the motion of particles in a rotating frame.

**Transformations**

Creating a transformation to a rotating frame from an inertial frame (where the rotating axis is at rest) means the following constraints.

- \( \Delta \theta_r = 0 \) if \( \Delta \theta_i = \omega \Delta t \).
- The inertial observer, relative to the rotating frame, must rotate at angular frequency \(-\omega\). This is to satisfy the obvious motion of stars relative to the rotating Earth frame.

Satisfying these constraints means assuming a form for the transformation

\[
\theta_r = a \cdot (\theta_i - \omega t_i) \quad r_r = r_i \quad z_r = z_i \quad t_r = at_i + \left( \frac{L}{c} \right)(d \cdot \theta_i)
\]  

(25)

The variables \( a \) and \( d \) are unit-less quantities. Using these to satisfy the condition that \( ds^2 \) for two nearby events must be the same for both observers (via numerical methods) gives

\[
a = 1 \quad d = 0
\]

(26)

so

\[
\theta_r = \theta_i - \omega t_i \quad r_r = r_i \quad z_r = z_i \quad t_r = t_i
\]

(27)

A rather Galilean looking transformation!

**Quantum wave equation**

Applying the rotating frame scale factors to quantum mechanics means the generalized Klein-Gordon equation\(^4\) becomes.

\[
-\hbar^2 \left( \frac{1}{m_0} \cdot \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \phi^2} \right) \Psi + \frac{\omega \gamma r}{E_0} \cdot \vec{p} \cdot \hat{E} \cdot \Psi = \frac{\gamma^2}{2E_0} \cdot \Psi - \frac{E_0}{2} \cdot \Psi
\]

(28)

The tildes signify momentum and total energy (relativistic and potential) operators.
Summary
A rotating frame can be viewed as a non-orthogonal reference frame and the path of an object acting under the Centripetal and Coriolis force is a geodesic of the object through this space-time. Moreover the velocity of a beam of light in vacuum is not in general an invariant, but can vary with the geometry.

Works Cited
4. Kevin Gibson *Generalizing the Klein-Gordon equation* (unpublished)
   http://www.mc.maricopa.edu/~kevinlg/i256/Generalized_KG.pdf
Contact information
kevin.gibson@asu.edu
kevinlg@mesacc.edu