Example

\[ f(x) = \sqrt{x} \]
\[ f'(x) = \frac{1}{2\sqrt{x}} \]

\[ f'(b) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \]

\[ f(a) \approx f(b) - f'(b)(b-a) \]
\[ \sqrt{3} \approx 2 - \frac{1}{4}(4-3) \]
\[ \approx 2 - \frac{1}{4}(1) \]
\[ \approx 1.75 \]

Two methods to solve linear approximation problems:
1) Find a tangent then plug in values
2) Use the formula (don't use if you have to solve within a certain margin of error)

CURVE SKETCHING

Another application of the derivative.

\[ f'(x) = 0 \text{ is a critical point} \]

Finding critical points using tangents is called "first derivative analysis."

- Derivatives in this area are positive:
  - \( f'(x) = + \)
  - \( f(x) \) is increasing

- Derivatives in this area are negative:
  - \( f'(x) = - \)
  - \( f(x) \) is decreasing
Note: When \( f'(x) = 0 \), a critical point is not guaranteed.

Critical points may exist when
1) \( f'(x) = 0 \)
2) \( f'(x) \) does not exist

When is the first derivative positive or negative?
Use factor analysis.

Example:
\[
y = -3x^5 + 5x^3
\]
\[
\frac{dy}{dx} = -15x^4 + 15x^2
\]
\[
= -15x^2(x^2 - 1)
\]
\[
= -15x^2(x - 1)(x + 1)
\]

Let \( 0 = -15x^2(x - 1)(x + 1) \)
\( x = 0, x = 1, x = -1 \)
These are possible critical values (or numbers).

\[
f(0) = -3(0)^5 + 5(0)^3 = 0 \quad (0, 0) \quad \text{these are}
\]
\[
f(1) = -3(1)^5 + 5(1)^3 = 2 \quad (1, 2) \quad \text{possible critical points.}
\]
\[
f(-1) = -3(-1)^5 + 5(-1)^3 = -2 \quad (-1, -2)
\]

The other possibility \((0, 0)\) is not a CP.
SECOND DERIVATIVE ANALYSIS

\[ f''(x) = \frac{d}{dx} \left( \frac{f'(x)}{L_{slope}} \right) \]

The second derivative is the rate of change of slopes of tangents.

Changes in slope from \( \Theta \) to \( \Theta \): concave down \( \downarrow \)
Changes in slope from \( \Theta \) to \( \Theta \): concave up \( \uparrow \)

When \( f''(x) = 0 \), there is a point of inflection.

**Example:**

\[ f''(x) = -60x^2 + 30x \]

\[ = -30x(2x^2 - 1) \]

\[ = -30x(\sqrt{2}x - 1)(\sqrt{2}x + 1) \]

\[ 0 = -30x(\sqrt{2}x - 1)(\sqrt{2}x + 1) \]

\[ x = 0, \quad x = \frac{1}{\sqrt{2}}, \quad x = -\frac{1}{\sqrt{2}} \]

Possible inflection values

\[ f(0) = -3(0)^5 + 5(0)^3 = 0 \quad (0,0) \]

\[ f\left(\frac{1}{\sqrt{2}}\right) = -3\left(\frac{1}{\sqrt{2}}\right)^5 + 5\left(\frac{1}{\sqrt{2}}\right)^3 = 1.23 \quad (0.707, 1.23) \]

\[ f\left(-\frac{1}{\sqrt{2}}\right) = -3\left(-\frac{1}{\sqrt{2}}\right)^5 + 5\left(-\frac{1}{\sqrt{2}}\right)^3 = -1.23 \quad (0.707, -1.23) \]

Possible inflection points

---

**Graph:**

Points of inflection (IP) and critical points (CP).
Four basic curves:

concave up

concave down

increasing

decreasing

example

\[ f(x) = x^{8/3} - 5x^{4/3} \]

let \( 0 = \frac{5(x-2)}{9x^{4/3}} \)

\[ 0 = 5(x-2) \quad 0 = 9x^{4/3} \]

\[ 0 = x-2 \quad 0 = x^{4/3} \]

\( x = 2 \quad x = 0 \)

\( y = -3 \sqrt[4]{q} \quad y = 0 \)

\( (2, -3 \sqrt[4]{q}) \quad (0, 0) \)

let \( 0 = \frac{10(x+1)}{9x^{4/3}} \)

let \( \frac{dnc}{\frac{10(x+1)}{9x^{4/3}}} \)

\( 0 = 10(x+1) \quad 0 = 9x^{4/3} \)

\( 0 = x+1 \quad 0 = x^{4/3} \)

\( x = -1 \quad x = 0 \)

\( y = -6 \quad y = 0 \)

\( (-1, -6) \quad (0, 0) \)

Both possible CP's are CP's.

Only \((-1, -6)\) is an IP.
A vertical asymptote can act on a function just like a critical point, or an inflection point.

Example:

\[ f(x) = \frac{x^2+1}{x^2-4} \]

Vertical Asymptote (VA): \(-2, 2\)

Horizontal Asymptote (HA): \(y = 1\)

\[ f'(x) = 0 \quad \frac{-10x}{(x^2-4)^2} = 0 \]

\[ f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3} = 0 \]

Critical Points (CP):

Local Max (LMAX): \(x = \frac{2}{3}\)

Values of the Function:

\[ f(x) = 0 \quad \frac{10(x^2+1)}{(x^2-4)^3} = dne \]

Solutions:

\[ 3x^2 = -4 \quad (x-2)(x+2) = 0 \]

\[ x^2 = -\frac{4}{3} \quad x = 2, x = -2 \] (VA's)

No solution

Graph:

- HA: \(x = 1\)
- VA: \(x = \pm 2\)
- CP: \(LMAX, \left(\frac{2}{3}, -\frac{1}{4}\right)\)
Tools for curve sketching:
1) domain check
2) holes and breaks
3) asymptotes
4) x, y intercepts
5) crossovers
6) symmetry
7) transformations
8) factor analysis of f(x)
9) f'(x) analysis
10) f''(x) analysis

**MEAN VALUE THEOREM**

If f(x) is continuous and differentiable over the interval, there is at least one tangent at some point f'(c) which is parallel to the secant.

So \( m_T = m_s \)

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Example**

\( y = x^2 \) over \([1, 3]\). Find c.

Is \( x^2 \) continuous? YES

Is \( x^2 \) differentiable? YES

\[
\frac{dy}{dx} = 2x
\]

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

\[
2c = \frac{9 - 1}{3 - 1}
\]

\[
2c = \frac{8}{2}
\]

\[
c = 2
\]
L'Hospital's Rule

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x \to a} g(x) \neq 0 \text{ or } \pm \infty \]

Example

\[ \lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0} \]

\[ = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1 \]

This rule can be repeated over and over.

Example

\[ \lim_{x \to \infty} \frac{4x^2 + 6x}{3x^2 - 9x + 9} = \frac{\infty}{\infty} \]

\[ = \lim_{x \to \infty} \frac{8x + 6}{3x - 9} = \infty \]

\[ = \lim_{x \to \infty} \frac{8}{10} = \frac{8}{10} = \frac{4}{5} \]

Proof

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x \to a} g(x) \neq 0 \text{ or } \pm \infty \]

Now

\[ \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(a) - f(x)}{g(a) - g(x)} \]

\[ = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \]

\[ = \lim_{x \to a} \frac{f(x)}{g(x)} - \lim_{x \to a} \frac{f(a)}{g(x)} \]

Example

\[ \lim_{x \to \infty} e^{-x} \sqrt{x} = 0 \cdot \infty \]

\[ = \lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \infty \]

\[ = \lim_{x \to \infty} \frac{\sqrt{2x - \frac{1}{2}}}{e^x} \]

\[ = \lim_{x \to \infty} \frac{1}{2e^{-\sqrt{x}}} = \frac{1}{\infty} = 0 \]

To solve limits of the other indeterminate forms (not 0 or \( \infty \)), convert them to 0 or \( \frac{\infty}{\infty} \).
\[ \lim_{x \to 1^+} \frac{1}{x} - \frac{1}{x-1} = \infty - \infty \]
\[ = \lim_{x \to 1^+} \frac{1 - \ln x}{x-1} = 0 \]
\[ = \lim_{x \to 1^+} \frac{\ln x}{x-1} \]
\[ = \lim_{x \to 1^+} \frac{\ln x + 1 - \frac{1}{x}}{1} = 0 \]
\[ = \lim_{x \to 1^+} \frac{1}{2 + \ln x} = \frac{1}{2} \]

\[ \lim_{x \to 0^+} \left(1 + x\right)^{1/x} = \left(1 + 0\right)^{1/0} = 1^\infty = 1 \]

\[ y = \lim_{x \to 0^+} \left(1 + x\right)^{1/x} \]
\[ \ln y = \lim_{x \to 0^+} \ln \left(1 + x\right)^{1/x} = \lim_{x \to 0^+} \frac{\ln \left(1 + x\right)}{x} \]
\[ = \lim_{x \to 0^+} \frac{\ln \left(1 + x\right)}{1} = 1 \]
\[ e^y = y \]
\[ y = e \]

\[ \lim_{x \to 0^+} (\sin x)^x = 0^0 \]
\[ y = \lim_{x \to 0^+} (\sin x)^x \]
\[ \ln y = \lim_{x \to 0^+} \ln (\sin x)^x = \lim_{x \to 0^+} x \ln (\sin x) \]
\[ = \lim_{x \to 0^+} \frac{\ln (\sin x)}{1/x} = \frac{\sin x}{0} \]
\[ = \lim_{x \to 0^+} \frac{\cos x / \sin x}{-1/x^2} = 0 \]
\[ = \lim_{x \to 0^+} \frac{\sin x / \cos x}{0} = 0 \]
\[ = \lim_{x \to 0^+} \frac{\sin x + \cos x (2x)}{\cos x} = 0 \]
\[ y = e^0 \]
\[ y = 1 \]
Example
\[
\lim_{x \to \infty} x^{1/x} = \infty
\]
\[
y = \lim_{x \to \infty} x^{1/x}
\]
\[
\ln y = \lim_{x \to \infty} \ln x^{1/x} = \lim_{x \to \infty} \frac{\ln x}{x}
\]
\[
= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 = \ln y
\]
\[
y = e^0
\]
\[
y = 1
\]

Story Problems
In these problems, always look for what's being maximized or minimized.

Example

Fence available = 2400 ft

This is called a "constraint".

What dimensions will create the maximum area?

Objective: \( A = xy \)  
Constraint: \( 2400 = 2x + y \)

\[ A = x(2400 - 2x) \]
\[ 2400 - 2x = y \]
\[ \frac{dA}{dx} = 2400 - 4x \]

Let \( 0 = 2400 - 4x \)  
\[ 2400 - 2(600) = y \]
\[ 4x = 2400 \]
\[ y = 1200 \]

\( x = 600 \)  
One possible max (at CP)

We still need to check if one of the end points for the absolute maximum.

Method One:

First derivative analysis:

\[
\begin{array}{cccccc}
\text{end} & \text{CP} & \text{end} & \text{point} & \text{point} & \text{point} \\
\text{point} & x = 0 & \text{LIMAX} & \text{600 - x} & 600 & x = 400 \\
\text{y = 1200} & \text{720,000} & \end{array}
\]
METHOD TWO

Second derivative analysis
\[ f''(x) = -4 \]
\[ \therefore \text{Always concave down, so } \text{CP} \rightarrow \text{Absolute max.} \]

METHOD THREE

Evaluate critical and end points

\[ \begin{array}{c|c|c}
\text{X} & \text{A} \\
\hline
\text{EP} & 0 & 0 \\
\text{CP} & 600 & 720,000 \quad \text{Absolute max.} \\
\text{EP} & 1200 & 0 \\
\end{array} \]

**ECONOMICS**

Cost = Fixed cost + Variable cost
(e.g. rent, utilities) (e.g. materials)

\[ \text{Cost} - C = cx \quad (c = \text{cost per unit}, \ x = \# \text{ of units}) \]

Revenue \[ R = rx \quad (r = \text{revenue per unit}) \]

Profit \[ P = px \quad (p = \text{profit per unit}) \]

Marginal cost = \[ \frac{dc}{dx} \]
Marginal revenue = \[ \frac{dR}{dx} \]
Marginal profit = \[ \frac{dp}{dx} \]

**Example**

<table>
<thead>
<tr>
<th>X</th>
<th>r</th>
<th>( r - r_i = m(x-x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>$450/unit</td>
<td>( r = -\frac{1}{10} (x-1000) + 450 )</td>
</tr>
<tr>
<td>1100</td>
<td>$410/unit</td>
<td>( r = -\frac{1}{10} x + 100 + 450 )</td>
</tr>
<tr>
<td>1200</td>
<td>$430/unit</td>
<td>( r = -\frac{1}{10} x + 550 )</td>
</tr>
</tbody>
</table>

\[ m = \frac{x_2-x_1}{y_2-y_1} = \frac{450-410}{1000-1100} = \frac{-50}{-100} = \frac{1}{10} \]

Price of unit for any \( x \)
What price will maximize revenue?

\[ P = r_0 x \]

\[ R = \left(-\frac{1}{10}x + 550\right)x \]

\[ R = \frac{1}{10} x^2 + 550x \]

\[ \frac{dR}{dx} = \frac{1}{10}(2x) + 550 \]

\[ \frac{dR}{dx} = \frac{1}{5} x + 550 \]

let \( \frac{dR}{dx} = 0 \)

\[ \frac{1}{5} x + 550 = 0 \]

\[ x = 2,750 \text{ units} \]

How much of a rebate from original price will maximize revenue?

\[ r(2750) = \frac{1}{10}(2750) + 550 \]

\[ = 275 + 550 \]

\[ = $2,750 \]

\[ 450 - 275 = $175 \text{ rebate} \]

The weekly cost is \( C(x) = \frac{68,000 + 150x}{\text{fixed}} \)

\[ \text{variable} \]

What should the rebate be to maximize profit?

\[ P = R - C \]

\[ P = \left(-\frac{1}{10}x^2 + 550x\right) - \left(68,000 + 150x\right) \]

\[ P = \frac{1}{10} x^2 + 400x - 68,000 \]

\[ \frac{dP}{dx} = \frac{1}{10} x + 400 \]

let \( \frac{dP}{dx} = 0 \)

\[ x = 2,000 \text{ units} \]

\[ P(2000) = \frac{1}{10} (2000)^2 + 400 \]

\[ = 4,000 \]

\[ 1,000 - 4,000 = $332,000 \text{ max profit} \]

\[ r = \frac{1}{10} (2000) + 550 \]

\[ = $350/\text{unit} \]

\[ 450 - 350 = $100 \text{ rebate} \]
NEWTON'S METHOD
A method of approximation which finds the zeros (roots or x-intercepts) of functions.

Steps:
1) guess a solution, e.g. $x_0$
2) find the tangent at that point $f'(x_0)$
   
   so \[ y_i - y_0 = m(x_i - x_0) \]
   
   \[ y_i - f(x_0) = f'(x_0)(x_i - x_0) \]
   
   \[ y = f'(x_0)(x_i - x_0) + f(x_0) \]

   This is a linear approximation.

3) Make a better guess using the root of the tangent.
   
   \[ x = f'(x_0)(x_i - x_0) + f(x_0) \]
   
   \[ f(x) = f'(x_0)(x_i - x_0) \]
   
   \[ \frac{f(x_0)}{f'(x_0)} = x_i - x_0 \]
   
   \[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

4) Repeat for a new, even better guess

   \[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]
   
   \[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \]

   \[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Note: The calculator's solver function uses Newton's method.

Example:
Find a root of $x^2 - 5x + 6 = (x - 3)(x - 2)$

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$f(x_n)$</th>
<th>$f'(x_n)$</th>
<th>$x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$</th>
</tr>
</thead>
</table>
| guess | $1^2 - 5(1) + 6 = 2$ | $2 - 5$ | \[x_{1+1} = x_1 - \frac{2}{2} = 1 - \frac{2}{2} = 1.000007 \]
| 1.4667 | 0.4444 | -1.6666 | 1.93333 |
| 1.7333 | 0.0711 | -1.1333 | 1.99407 |
| 1.9940 | 0.0035 | -1.0078 | 1.99999 |
| 1.99999 | 0.00001 | -1.00002 | 2 | one of the roots
Newton's method works only for real roots.

**Example**

\[ \sqrt{13} = ? \]

\[ (x - \sqrt{13}) = 0 \]

\[ x = \sqrt{13} \]

\[ x^2 = 13 \]

\[ x^2 - 13 = 0 \quad f(x) = x^2 - 13 \]

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>( f(x_n) = x^2 - 13 )</th>
<th>( f'(x_n) = 2x )</th>
<th>( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>guess 3</td>
<td>-4</td>
<td>6</td>
<td>3.66667</td>
</tr>
<tr>
<td>3.66667</td>
<td>0.44447</td>
<td>7.33334</td>
<td>3.40406</td>
</tr>
<tr>
<td>3.40406</td>
<td>0.06348</td>
<td>7.21212</td>
<td>3.40555</td>
</tr>
<tr>
<td>3.40555</td>
<td>0.00009</td>
<td>7.21110</td>
<td>3.40554</td>
</tr>
</tbody>
</table>

This value should approach zero.

**Example**

where do \( y = x^3 + x^2 - 5 \)

\[ x^3 + x^2 - 5 = x^2 + 2x \]

\[ y = x^2 + 2x \] intersect? \( x^3 - 2x - 5 = 0 \)

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>guess 2</td>
<td>-1</td>
<td>10</td>
<td>2.1</td>
</tr>
<tr>
<td>2.1</td>
<td>0.061</td>
<td>11.23</td>
<td>2.0946</td>
</tr>
<tr>
<td>2.0946</td>
<td>0.000018</td>
<td>11.16</td>
<td>2.0946</td>
</tr>
</tbody>
</table>

\[ @ (2.0946, 8.5765) \]

Problems with Newton's method —

**#1**

First guess \( x_0 \) leads to \( x_1 \), which leads to \( x_0 \), which leads to \( x_1 \), etc.

To avoid this problem, pick a different first guess. (Keep an eye on \( f(x_n) \) to make sure it's going to 0.)

**#2** Method breaks down if \( f(x) \) is not differentiable at \( x_0, x_1, x_2 \ldots \)